

# On Delusive Nodes of Free Oscillations

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**In memoriam of Vladimir Arnold**

## Abstract

Two theorems and one conjecture about nodal sets of eigenfunctions arising in various spectral problems for the Laplacian are reviewed. It occurred that all these assertions are incorrect or only partly correct, but their analysis has brought better understanding of the corresponding area of mathematical physics. The contribution made by V.I. Arnold is emphasized.

The name of Vladimir Arnold, who passed away on 3 June 2010, is well known to mathematicians all over the world. Indeed, along with the Kolmogorov–Arnold–Moser theory about the stability of integrable systems (his best known contribution to mathematics) there are several other notions associated with him, for example: the Arnold conjecture about the number of fixed points that a smooth function has on a closed manifold, Arnold's cat map, the Arnold diffusion, an Arnold tongue in dynamical systems theory.

A biographical sketch of Vladimir Igorevich Arnold by O'Connor and Robertson (an entry of MacTutor History of Mathematics archive) is available online at <http://www-history.mcs.st-andrews.ac.uk/Biographies/Arnold.html>. A lot of interesting details about Arnold's life and work are presented by his colleagues and disciples in the tribute and memories published in 2012; see [21] and [22], respectively. From these notes one gets a clear idea that everybody, who maintained contact with him, was greatly impressed by his extraordinary personality.

Among Arnold's numerous honours one finds Dannie Heineman Prize for Mathematical Physics awarded in 2001 jointly by the American Physical Society and American Institute of Physics. This is not an incident because he had a deep feeling of the unity of mathematics and natural sciences. His often quoted remark says that mathematics is a part of physics in which experiments are cheap.

No wonder that one of Arnold's papers published posthumously deals with an important property of eigenoscillations in Mathematical Physics; see [3], submitted for publication six months before his death. In this paper<sup>0</sup>, Arnold with his inherent mastery of both the subject and storytelling, describes a fascinating fact that an incorrect theorem was announced in the classical book [8] by Courant and Hilbert.

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<sup>0</sup>An item in the collection dedicated to the 75th anniversary of the Steklov Mathematical Institute in Moscow. Before 1934, when the Soviet Academy of Sciences was moved from Leningrad to Moscow, this institute was a division of the Physical–Mathematical Institute organised by V.A. Steklov in 1921 (see Steklov's recollections cited in [25]).



Vladimir Igorevich Arnold in 1977.

(This edition is cited in [3], but, for the reason explained below, Arnold used either the 2nd German edition [9] or, most likely, its Russian translation published twice in 1933 and 1951.)

The theorem in question deals with nodal sets (or, for brevity, nodes) of linear combinations of some particular eigenfunctions (see the next paragraph). Such a set is simply defined as the set, where a function vanishes. To make the importance of eigenfunctions clear, we just mention that they serve for describing free oscillations of strings and membranes and nodes show, where an oscillating object is immovable because, by its definition a node separates the sets, where the function is positive and negative. In one, two and three dimensions, nodal sets consist of points, curves and surfaces, respectively. Pictures of nodal curves for some modes of oscillations of the square membrane fixed along its boundary can be found in many textbooks (see, for example, [36], p. 266).

It is amazing that there are rather many theorems and conjectures proved to be incorrect in this area of research. Let us list those considered in this paper and recall other renowned questions concerning the same spectral problems of mathematical physics. We begin with the theorem which is the topic of the Arnold's paper [3]. It concerns nodes of linear combinations of eigenfunctions of the Dirichlet Laplacian and we illustrate the question's essence with some elementary examples. This material is presented in the first section.

What is widely known about the eigenvalue problem for the Dirichlet Laplacian is the question 'Can one hear the shape of a drum?' posed by Mark Kac in 1966 in the title of his paper [20]. However, this question is about the whole set of eigenvalues, whereas there are many subtle questions about properties of eigenfunctions corresponding to individual eigenvalues. One of them, referred to as Payne's conjecture, concerns nodes of the second eigenfunction; being more technical, it is considered in the third section.

It is worth mentioning that the negative answer to the Kac's question was obtained in 1992; it is presented in the form accessible to a general audience in the

article [14]. However, this answer, like falsity of the above mentioned theorem discussed in [3], is only a part of the story. In November 2012, S. Titarenko presented another part at the Smirnov Seminar on Mathematical Physics in St. Petersburg ([http://www.pdmi.ras.ru/~matfizik/seminar\\_2012-2013.htm](http://www.pdmi.ras.ru/~matfizik/seminar_2012-2013.htm)). The most important point of his talk entitled ‘When can one hear the shape of a drum? Sufficient conditions’ is that for answering in the positive to the Kac’s question the boundary of drum’s membrane must be smooth. Indeed, smoothness is violated in all of the now numerous examples delivering the negative answer (see, for example, [13], p. 2235; this article also contains an extensive list of references on mathematical and physical aspects of isospectrality). Unfortunately, Titarenko’s result is still unpublished.

The second section deals with the well known phenomenon of liquid sloshing in containers (widely used examples of these are tea cups, coffee mugs, wine glasses, cognac snifters *etc*). The corresponding mathematical model — the so-called sloshing problem (it is also referred to as the mixed Steklov problem) — attracted much attention after awarding the 2012 Ig Nobel Prize for Fluid Dynamics to R. Krechetnikov and H. Mayer for their investigation why coffee so often spills while people walk with a filled mug [30]. This effect results from the correlation between the fundamental sloshing frequency and that of steps. Here, a property of the sloshing nodes (the liquid remains immovable there during its free oscillations) is considered. The presented example demonstrates that a gap in the proof of a certain theorem describing the behaviour of nodes cannot be filled up.

Another aim of this paper is to show how application of rather simple tools (in particular, an analysis of the behaviour of functions defined explicitly, for example, by improper integrals and even by elementary trigonometrical formulae) leads to interesting results concerning important questions that challenge both mathematical and physical intuition. It should be emphasised that such questions were among Arnold’s favorites. Indeed, his unique intuition, for example, in the subject of catastrophes allowed him to guess on the spot the right answers when physicists and engineers asked him what kind of catastrophic effect could be expected in their problems. Many of his guesses were based on very simple models like that considered in the next section.

## Arnold on a footnote in the Courant–Hilbert book

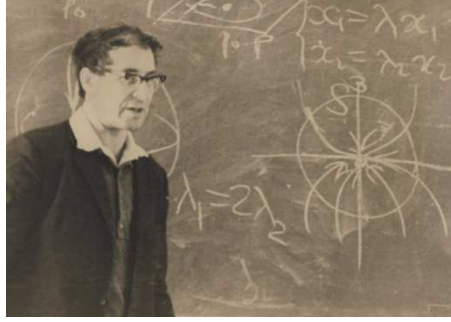
Arnold begins his story with the following

topological result [...] valid on any compact manifold: an eigenfunction  $u$  of the Laplace operator

$$\Delta u = \lambda u \quad \text{with eigenvalue } \lambda = \lambda_n$$

(we arrange them in order of increasing frequencies  $-\lambda_1 \leq -\lambda_2 \leq -\lambda_3 \leq \dots$ ) vanishes on the oscillating manifold  $M$  in a way such that its zeros divide  $M$  into at most  $n$  parts.

In its original form, the result obtained by Courant in 1923 concerns nodes of eigenfunctions of a self-adjoint second order differential operator (for example, the Sturm–Liouville operator on an interval and the Laplacian in a bounded higher-dimensional domain) with one of the standard boundary conditions (for example, the Dirichlet and Neumann conditions). Namely, Courant’s theorem asserts that (see [8], p. 452):



V. Arnold lecturing in Syktyvkar in 1977.

if [the] eigenfunctions are ordered according to increasing eigenvalues, then the nodes of the  $n$ th eigenfunction divide the domain into no more than  $n$  subdomains. No assumptions are made about the number of independent variables.

Two simplest examples illustrating this theorem are delivered by the equation describing the set of possible shapes of an homogeneous string in free time-harmonic oscillations:

$$-u'' = \lambda u \quad \text{on } (0, \pi), \quad (1)$$

augmented by either the Dirichlet conditions

$$u(0) = u(\pi) = 0, \quad (2)$$

which means that the ends of a string are fixed, or the Neumann conditions

$$u'(0) = u'(\pi) = 0 \quad (3)$$

when the ends are free. It is clear that the eigenfunction  $u_n = \sin nx$ ,  $n = 1, 2, \dots$ , corresponds to  $\lambda_n = n^2$  under the boundary conditions (2), whereas conditions (3) give

$$u_n = \cos(n-1)x \quad \text{and} \quad \lambda_n = (n-1)^2, \quad \text{respectively.}$$

Note that in both cases the  $n$ th eigenfunction divides the interval into precisely  $n$  parts. Courant proves that this property remains valid for a general Sturm–Liouville problem.

Prior to proving the latter result, a footnote announcing the notorious incorrect theorem appears at the end of the proof of the theorem cited above (see the first footnote on p. 454 in [8]):

The theorem just proved may be generalized as follows: Any linear combination of the first  $n$  eigenfunctions divides the domain, by means of its nodes, into no more than  $n$  subdomains. See the Göttingen dissertation of H. Herrmann, Beiträge zur Theorie der Eigenwerte und Eigenfunctionen, 1932.

Below, this assertion is referred to as Herrmann’s theorem. Arnold writes about it:

This *generalization of Courants theorem* is not proved at all in the book by Courant and Hilbert; it was just mentioned that the proof “will soon be published (by a disciple of Courant)”.

From the last sentence we see that Arnold used either the 2nd German edition [9] published in 1931 or, what is more likely, its Russian translation. Then he continues:

Having read all this, I wrote a letter to Courant, “Where can I find this proof now, 40 years after Courant announced the theorem?” Courant answered that “one can never trust one’s students: to any question they answer either that the problem is too easy to waste time on, or that it is beyond their weak powers”.

As regards Courant and Hilbert’s *Mathematical Physics*, according to Courant’s published recollections, this book was nevertheless written by his students.

Of course, Arnold exaggerates the role of students, but at the beginning of preface to [8] Courant writes that the second German edition was “revised and improved with the help of K. O. Friedrichs, R. Luneburg, F. Rellich, and other unselfish friends”.

Soon after receiving Courant’s reply, Arnold discovered that applying Herrmann’s theorem to the eigenfunctions of the Laplacian on the sphere  $S^N$  with the standard Riemannian metric one obtains an estimate for the number of components complementing a real algebraic hypersurface of the degree  $n$  in the  $N$ -dimensional projective space (see [4]). The idea behind this is that the so-called spherical harmonics (eigenfunctions of the Laplacian on the two-dimensional sphere) are defined as follows. The set of these functions corresponding to the  $n$ th eigenvalue consists of traces on  $S^2$  of homogeneous harmonic polynomials of the degree  $n - 1$  in  $\mathbb{R}^3$  (see [36], p. 263). Hence a linear combination of eigenfunctions corresponding to the first  $n$  eigenvalues is also a harmonic polynomial whose degree is bounded by  $n$ . In [3], Arnold comments his estimate as follows:

[...] it turned out that the results of the topology of algebraic curves that I had derived from the generalized Courant theorem contradict the results of quantum field theory. Nevertheless, I knew that both my results and the results of quantum field theory were true. Hence, the statement of the generalized Courant theorem is not true (explicit counterexamples were soon produced by Viro). Courant died in 1972 and could not have known about this counterexample.

Indeed, seven years after Courant’s death, Viro found an example of real algebraic hypersurface for which Arnold’s estimate does not hold, thus establishing what is incorrect about Herrmann’s theorem. Namely, it is valid only under some restrictions on the number of independent variables, in particular, it is false for the Laplacian on  $S^3$  and higher-dimensional spheres (see [37]).

However, Herrmann’s theorem is true for eigenfunctions of the Dirichlet and Neumann problems for equation (1). This follows from elementary trigonometric formulae (see 1.331.1 and 1331.3 in [15]). Indeed, if  $n > 1$ , then the  $n$ th Dirichlet and Neumann eigenfunctions can be written as follows:

$$\sin nx = \sin x \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n-k-1}{k} (2 \cos x)^{n-(2k+1)}, \quad (4)$$

$$\begin{aligned} \cos(n-1)x &= 2^{n-2} \cos^{n-1} x \\ &+ \frac{n-1}{2} \sum_{k=1}^{[(n-1)/2]} \frac{(-1)^k}{k} \binom{n-k-2}{k-1} (2 \cos x)^{n-(2k+1)}. \end{aligned} \quad (5)$$

Here  $[m]$  stands for the integer part of  $m$ .

According to formula (4), a linear combination of the first  $n$  Dirichlet eigenfunctions is the product of  $\sin x$  and a polynomial of  $\cos x$  whose degree is at most  $n - 1$ . Therefore, it has at most  $n - 1$  zeros and the number of nodes on  $(0, \pi)$  is also less than or equal to  $n - 1$ . The similar conclusion follows from (5) for a linear combination of the first  $n$  Neumann eigenfunctions. Let us illustrate this considering linear combinations of the first two Dirichlet and Neumann eigenfunctions which are

$$\sin x(C_1 + 2C_2 \cos x) \text{ and } C_1 + C_2 \cos x, \text{ respectively.}$$

Here  $C_1$  and  $C_2$  are some constants. Both linear combinations have at most one node on  $(0, \pi)$ . It exists when  $C_2 \neq 0$  and also

$$\left| \frac{C_1}{C_2} \right| < 2 \text{ and } \left| \frac{C_1}{C_2} \right| < 1$$

for the combinations of the Dirichlet and Neumann eigenfunctions, respectively. These conditions are also necessary for the existence of a node.

In the same way, one obtains that Herrmann's theorem is true for eigenfunctions of the following problem:

$$-u'' = \lambda u \text{ on } (0, 2\pi), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi).$$

For this periodic boundary value problem we have:

- The 1st eigenvalue is zero; it is simple and the corresponding eigenfunction is a non-zero constant.
- The  $n$ th eigenvalue is  $(n - 1)^2$ ; its multiplicity is two and the corresponding eigenfunctions are  $\sin(n - 1)x$  and  $\cos(n - 1)x$ .

Then Herrmann's theorem is again a consequence of formulae (4) and (5), but  $n$  must be changed to  $n - 1$  in (4).

In the second section of [4], Arnold turns to the following Sturm–Liouville problem:

$$-u'' + qu = \lambda u \text{ on } (0, \ell), \quad u(0) = u(\ell) = 0, \quad (6)$$

here  $q$  is a positive function on  $[0, \ell]$ . He outlines Gel'fand's idea how to prove Herrmann's theorem for eigenfunctions of this problem. It consists in replacing

the analysis of the system of  $n$  eigenfunctions of the one-particle quantum-mechanical problem by the analysis of the first eigenfunction of the  $n$ -particle problem (considering, as particles, fermions rather than bosons).

This approach so attracted Arnold that he included Herrmann's theorem for eigenfunctions of problem (6) together with Gel'fand's hint into the 3rd Russian edition of his *Ordinary Differential Equations* (see Problem 9 on the list of supplementary problems at the end of [5]).

In [3] Arnold devotes two pages to some details of Gel'fand's analysis, but at the end he writes.

Unfortunately, the arguments above do not yet provide a proof for this generalized theorem: many facts are still to be proved. [...]

Gel'fand did not publish anything concerning this: he only told me that he hoped his students would correct [...] his theory. He pinned high hopes on V.B. Lidskii and A.G. Kostyuchenko. Viktor Borisovich Lidskii told

me that “he knows how to prove all this.” [...] Although [his] arguments look convincing, the lack of a published formal text with a rigorous proof of the Courant–Gel’fand theorem is still distressing.

This is still true, but there is a hope that Victor Kleptsyn (Institut de Recherche Mathématique de Rennes) will soon fill in this gap. In his unpublished manuscript, he not only provides a proof for all gaps remaining in the above approach, but also suggests an alternative one using the heat equation.

## On sloshing nodal curves

A particular case of the mixed Steklov eigenvalue problem gives the so-called sloshing frequencies and the corresponding wave modes, that is, the natural frequencies and modes of the free motion of water occupying a reservoir. When the latter is an infinitely long canal of uniform cross-section  $W$ , the two-dimensional problem arises. In this case, the boundary  $\partial W$  consists of  $F = \{|x| < a, y = 0\}$  and  $B = \partial W \setminus \bar{F}$  lying in the half-plane  $y < 0$ . The former is referred to as the *free surface* of water, whereas the latter is *canal’s bottom*.

The velocity potential  $u(x, y)$  with the time-harmonic factor removed must satisfy the following boundary value problem:

$$u_{xx} + u_{yy} = 0 \quad \text{in } W, \quad (7)$$

$$u_y = \lambda u \quad \text{on } F, \quad (8)$$

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } B. \quad (9)$$

Here  $\mathbf{n}$  denotes the exterior unit normal on  $B$  and  $\lambda = \omega^2/g$  is the spectral parameter to be found along with  $u$  ( $\omega$  is the radian frequency of the water oscillations and  $g$  is the acceleration due to gravity). In order to exclude the non-physical zero eigenvalue of (7)–(9), it is usual to augment the problem’s statement by the orthogonality condition

$$\int_F u \, dx = 0. \quad (10)$$

The condition on  $F$  is the Steklov boundary condition first introduced by Steklov in 1896, but the standard reference for the Steklov problem is the paper [35] published in 1902. Problem (7)–(10) and its three-dimensional version has been the subject of a great number of studies over more than two centuries; see [11] for a historical review, whereas early results are presented in the Lamb’s classical treatise *Hydrodynamics* [28].

It is well-known that this problem has a discrete spectrum, that is, an infinitely increasing sequence of positive eigenvalues of finite multiplicity (the latter is the number of different eigenfunctions corresponding to a particular value of  $\lambda$ ). The corresponding eigenfunctions  $u_n$ ,  $n = 1, 2, \dots$ , form a complete system in an appropriate Hilbert space. Unlike eigenfunctions of the Dirichlet and Neumann Laplacian, the first paper about properties of solutions to (7)–(10) had been published by Kuttler only in 1984 (see [24]). Since then, a number of interesting results concerning the so-called ‘high spots’ of sloshing eigenfunctions has appeared (see the recent review [27] aimed at lay readers).

The main result of [24] is analogous to Courant’s theorem. Namely, if the eigenfunctions are ordered according to increasing eigenvalues, then the nodes of the  $n$ th

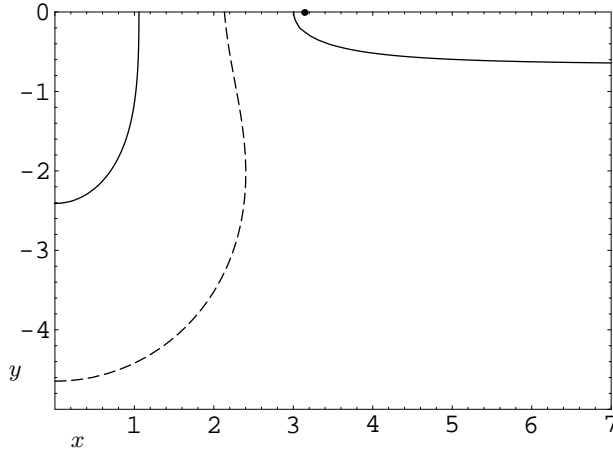


Figure 1: Nodal lines of  $u$  (solid lines) and  $v$  (dashed line) given by (11) and (12), respectively, with  $\nu = 3/2$ .

eigenfunction divide the domain into no more than  $n + 1$  subdomains. In view of the additional condition (10), the number of subdomains is  $n + 1$  instead of  $n$  appearing in Courant's theorem. Kuttler's reasoning (a version of Courant's original proof), indeed, proves this assertion after omitting the superfluous reference to the following incorrect lemma.

*For every eigenfunction of problem (7)–(10) nodal curves have one end on the free surface  $F$  and the other one on the bottom  $B$ .*

Counterexamples demonstrating that this lemma is incorrect were constructed twenty years after publication of [24]. They provide various domains  $W$  for which there exists an eigenfunction of problem (7)–(10) having a nodal curve with both ends on  $F$ . Let us outline the approach applied for this purpose in [23]. The example involves a particular pair velocity potential/stream function (the latter is a harmonic conjugate to the velocity potential) introduced in the book [26], § 4.1.1, namely,

$$u(x, y) = \int_0^\infty \frac{\cos k(x - \pi) + \cos k(x + \pi)}{k - \lambda} e^{ky} dk, \quad (11)$$

$$v(x, y) = \int_0^\infty \frac{\sin k(x - \pi) + \sin k(x + \pi)}{\lambda - k} e^{ky} dk, \quad (12)$$

where  $\lambda = m/2$  and  $m$  is odd. Then the numerators in both integrals vanish at  $k = \lambda$ , and so they are understood as usual infinite integrals. It is easy to verify that  $u$  and  $v$  are conjugate harmonic functions in the half-plane  $y < 0$ . Moreover, we have that

$$u(-x, y) = u(x, y) \quad \text{and} \quad v(-x, y) = -v(x, y), \quad (13)$$

which allows us to study the behaviour of nodal curves of these functions only in the quadrant  $\{x > 0, y < 0\}$  in view of their symmetry about the  $y$ -axis.

In § 2 of the paper [23], this behaviour is investigated in detail for  $\lambda = 3/2$  and illustrated in Fig. 1, where only the right half of the picture is shown in view of (13). It is proved that  $v$  has a nodal curve which has its both ends on the  $x$ -axis (dashed line). This nodal curve serves as  $B$  because the boundary condition (9) is fulfilled on it in view of the Cauchy–Riemann equations holding for  $u$  and  $v$ . Furthermore, there exists a nodal curve of  $u$  (solid line) lying in  $W$  defined by the described  $B$ .



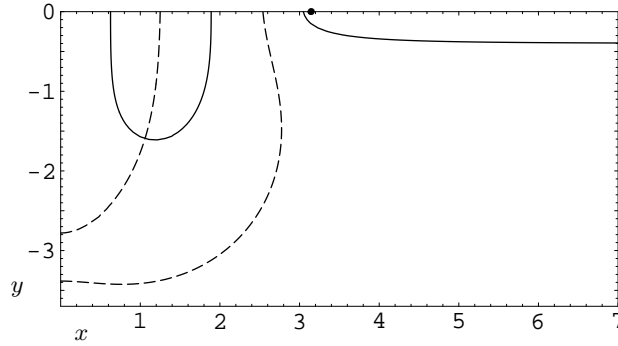


Figure 2: Nodal lines of  $u$  (solid lines) and  $v$  (dashed lines) given by (11) and (12), respectively, with  $\lambda = 5/2$ .

Moreover, it has both its ends on the  $x$ -axis, thus delivering a counterexample to Kuttler's lemma.

More complicated counterexamples to Kuttler's lemma are obtained numerically for  $\lambda = 5/2$ ; see Fig. 2, where again only the right half of the picture is shown. In this case, apart from the  $y$ -axis there are two nodes of  $v$  (dashed lines and their images in the  $y$ -axis) and four nodes of  $u$  (solid lines and their images in the  $y$ -axis). Both finite nodes of  $u$  are located within the domain  $W$  whose bottom  $B$  is given by the whole exterior node of  $v$ . In another counterexample, the bottom consists of the right half of this node complemented by the segment of the  $y$ -axis.

Besides, taking the interior node of  $v$  as the bottom, we see that the nodes of  $u$  connect this bottom with the corresponding free surface. Of course, the same is true for all known cases of the sloshing problem in two and three dimensions for which separation of variables is possible, thus providing a misleading hint.

## On nodal curves of oscillating membranes with fixed boundaries

The topic of this section is the eigenvalue problem

$$u_{xx} + u_{yy} + \lambda u = 0 \text{ in } D, \quad u = 0 \text{ on } \partial D, \quad (14)$$

where  $D$  is a bounded domain in  $\mathbb{R}^2$ . Its solutions  $(u_n, \lambda_n)$ ,  $n = 1, 2, \dots$  (for every  $\lambda > 0$  satisfying (14) the number of its repetitions is equal to its multiplicity) serve for representing pure tones that the elastic membrane  $D$  can produce being fixed along its boundary. As was mentioned above, along nodal curves an oscillating membrane stays immovable. This is why their study is important.

In the paper [16], published after defending his dissertation discussed above, Hermann remarked that Courant's theorem admit sharpening for eigenfunctions of problem (14). Such a refinement appeared in 1956 (see [33]), and is usually referred to as the Pleijel's nodal domain theorem nowadays. Its most interesting consequence says.

*The number of subdomains, into which the nodes of the  $k$ -th eigenfunction of problem (14) divide  $D$ , is equal to  $k$  only for finitely many values of  $k$ .*

In the last section of his note, Pleijel writes that "...it seems highly probable that the result [...] is also true for free membranes", that is, when the Dirichlet boundary condition is changed to the Neumann one in (14). This conjecture was recently proved by Polerovich [34] under the assumption that  $\partial D$  is piecewise analytic. The difficulty

of this case is that along with nodal subdomains lying totally in the interior of  $D$ , there are subdomains adjacent to  $\partial D$ , where the Neumann condition is imposed. To the former subdomains the original technique used by Pleijel and involving the Faber–Krahn isoperimetric inequality is applicable, whereas the latter ones require an alternative approach based on an estimate for the number of boundary zeros of Neumann eigenfunctions.

According to Courant’s theorem, the fundamental eigenfunction  $u_1$  does not change sign in  $D$ , whereas the node of  $u_2$  divides  $D$  into two subdomains. Both these cases give the maximal number of subdomains in a trivial way. Less trivial fact obtained in [33] is that only the first, second and fourth eigenfunctions give the maximal number of subdomains for a square membrane with fixed boundary.

During the past several decades, much attention was paid to the following question. *How does the only node of  $u_2$  divide  $D$  into two subdomains?* In his widely cited survey paper [31] published in 1967, Payne conjectured that the nodal curve of  $u_2$  cannot be closed for any domain  $D$  (see Conjecture 5 on p. 467 of his paper)<sup>0</sup>. It occurred that like Herrmann’s theorem this conjecture is only partly true. The corresponding results are outlined below.

Six years later, Payne proved the following theorem confirming his conjecture (see [32]).

*If  $D$  is convex in  $x$  and symmetric about the  $y$ -axis, then  $u_2$  cannot have an interior closed nodal curve.*

Prior to proving this assertion, Payne lists some important facts about eigenvalues and nodes of eigenfunctions that follow from the theory of elliptic equations. (In particular, it yields that all solutions of (14) are real analytic functions in the interior of  $D$ .) These properties are as follows:

- (i) If  $D'$  is strictly contained in  $D$ , then the inequality  $\lambda'_n > \lambda_n$  holds for the corresponding eigenvalues.
- (ii) No nodal curve can terminate in  $D$ .
- (iii) If two nodal curves have a common interior point, then they are transversal; this also applies when a nodal curve intersects itself.

Several partial results followed the above Payne’s theorem (see references cited in [2]) before Melas [29] proved that the conjecture is true for all convex two-dimensional domains with  $C^\infty$  boundary. This happened 25 year after it had been formulated. Two years later, this result was extended by Alessandrini to the case of general convex domains in  $\mathbb{R}^2$ . Namely, his theorem is as follows (see [2]).

*Let  $D$  be a bounded convex domain in the plane. If  $u$  is an eigenfunction corresponding to the second eigenvalue of problem (14), then the nodal curve of  $u$  intersects  $\partial D$  at exactly two points.*

Besides, Payne’s conjecture is also true for a class of non-convex planar domains as was recently shown in [38].

Let us turn to results demonstrating that Payne’s conjecture is not true for *all* bounded domains to say nothing of unbounded ones. The first counterexample to the general conjecture in  $\mathbb{R}^2$  belongs to M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof

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<sup>0</sup>It is worth mentioning that Yau repeated this question 15 years later, but only for convex plane domains. May be he expected that it is not true in its full generality.

and N. Nadirashvili [17] (see also [18]), who constructed a multiply connected domain such that the nodal set of  $u_2$  is disjoint with  $\partial D$ .

To describe such a domain we apply non-dimensional variables which is natural from the physical point of view remembering Arnold's remark about mathematics as a part of physics. Since the boundary of a domain considered in [17] involves two concentric circumferences (the origin is chosen to be their centre), we take the radius of the smaller circumference as the length unity. According to [17], the radius of the larger circumference, say,  $r \in (1, +\infty)$  must be taken so that the fundamental eigenvalue of problem (14) in the annulus with interior and exterior radii equal to 1 and  $r$ , respectively, lies strictly between the first and second eigenvalues of problem (14) in the unit circle. These values are well known being equal to  $j_{0,1}^2$  and  $j_{1,1}^2$ , respectively; here  $j_{0,1} \approx 2.405$  and  $j_{1,1} \approx 3.832$  are the least positive zeros of the Bessel functions  $J_0$  and  $J_1$ , respectively.

The standard separation of variables gives the fundamental eigenvalue for the described annulus. It is equal to  $\mu^2$ , where  $\mu(r)$  is the least positive root of the following equation

$$J_0(\lambda)Y_0(\lambda r) - J_0(\lambda r)Y_0(\lambda) = 0.$$

Here  $Y_0$  is the zero-order Bessel function of the second kind. Thus, the condition imposed on  $r$  can be written in the form:

$$2.405 \approx j_{0,1} < \mu(r) < j_{1,1} \approx 3.832. \quad (15)$$

The existence of  $r$  such that (15) is valid is considered by the authors of [17] as an obvious fact and its natural explanation from the physical point of view is as follows. Since  $\mu(r)$  is the frequency of free oscillations of an annulus with fixed boundary, it monotonically decreases from infinity to zero as the annulus width  $r - 1$  increases from zero to infinity, and so inequality (15) holds when  $r$  belongs to some intermediate interval. However, it is worth to give a quantitative evaluation of this interval and this can be easily done with the help of classical handbooks. The table on p. 204 in [19] gives that 2 belongs to this interval because  $\mu(2) \approx 3.123$ , whereas Table 9.7 in [1] shows that  $5/3$  and  $5/2$  are out of it because  $\mu(5/3) \approx 4.697$  and  $\mu(5/2) \approx 2.073$ . More detailed information about the behaviour of  $\mu(r)$  one gets from the graph plotted in figure 110 on p. 204 in [19].

The next step is characterised in [18] as "carving"  $N > 2$  holes in the circumference separating the unit circle from the annulus in order to obtain a single multiply connected domain; the angular diameter of each hole is  $2\epsilon$ , where  $\epsilon \in (0, \pi/N)$ . Therefore, it is convenient to use polar coordinates for this purpose:  $\rho \geq 0$  and  $\theta \in (-\pi, \pi]$  such that  $x = \rho \cos \theta$ ,  $y = \rho \sin \theta$ . The boundary of the sought domain  $D_{N,\epsilon}$  is as follows:

$$\partial D_{N,\epsilon} = \{\rho = r\} \cup \left\{ \rho = 1, \theta \notin \bigcup_{k=0}^{N-1} \left( \frac{2\pi k}{N} - \epsilon, \frac{2\pi k}{N} + \epsilon \right) \right\},$$

and so  $\partial D_{N,\epsilon}$  consists of  $N + 1$  (at least three) components.

Now we are in a position to formulate the main result proven in [17] and [18].

*Let  $r > 1$  be such that inequality (15) holds. Then there exists  $N_0 \geq 2$  such that for  $N \geq N_0$  and sufficiently small  $\epsilon = \epsilon(N)$  the following assertions are true: (i) the 2nd eigenvalue of problem (14) in the domain  $D_{N,\epsilon}$  is simple; (ii) the nodal curve of the corresponding eigenfunction  $u_2$  is a closed curve in  $D_{N,\epsilon}$ .*

In their proof, the authors use the symmetry of the domain  $D_{N,\epsilon}$ . Moreover, they note we have not tried to get an explicit bound on the constant  $N_0$  [...]. This [...] would probably lead to an astronomical number.

Then they conjecture that no simply connected domain has a closed nodal curve of  $u_2$ .

In 2001, Fournais [10] obtained “a natural higher dimensional generalisation of the domain” constructed in [17]. Instead of using the symmetry of a domain he applied an alternative, and in a some sense more direct, approach to “carving” evenly distributed holes in the inner sphere in order to obtain the desired conclusion.

The next step was to consider unbounded domains. In this case, Payne’s conjecture does not hold even for planar domains satisfying conditions used by Payne himself when proving the conjecture for bounded domains. Namely, the following theorem was obtained in [12].

*There exists a simply connected unbounded planar domain which is convex and symmetric with respect to two orthogonal directions, and for which the nodal line of a 2nd eigenfunction does not touch the domain’s boundary.*

## Brief conclusions

The above examples are taken from a rather narrow area in mathematical physics. Nevertheless, they clearly show that even incorrect and/or partly correct theorems and conjectures often lead to better understanding not only of the corresponding mathematical topic, but, sometimes, a topic in a completely distinct field.

Another conclusion concerns the role of style in Arnold’s papers and, especially, his books. It combines clarity of exposition, mathematical logic, physical intuition and masterly use of pictures. Therefore, it is not surprising that he is among the world’s most cited authors and No. 1 in Russia according to [http://www.mathnet.ru/php/person.phtml?&option\\_lang=eng](http://www.mathnet.ru/php/person.phtml?&option_lang=eng). Every mathematician would enjoy his papers aimed at a general audience; in particular, [6] and [7], which show that his English is as excellent as his Russian. Unfortunately, some translations of his papers leave a lot to be desired (for example, one finds ‘knots’ instead of ‘nodes’ in [3]; see the top paragraph on p. 26).

There is a common opinion that Agatha Christie’s novels are helpful for learning English (the author’s own experience confirms this). In much the same way, Arnold’s papers and books are helpful for both learning mathematics and learning to write mathematics.

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